

# Revisiting Absorbing Phase Transition in Energy Exchange Models

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A recent study of conserved Manna model, with both discrete and continuous variable, indicates that absorbing phase transitions therein belong to the directed percolation (DP) universality class. In this context we revisit critical behaviour in energy exchange models with a threshold. Contrary to the previous claims [*arXiv:1102.1631*, PRE **83**, 061130 (2011)], our results indicate that both the maximal and minimal versions of this model belong to the DP class.

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Absorbing state transitions (APT) [1, 2] in presence of a conserved field [3, 4] has drawn a lot of attention in the past decade. It is believed that critical behaviour of these systems is different from directed percolation (DP) [1], the generic universality class of absorbing state phase transitions. The most important of these systems is probably Manna model [5]; though originally introduced in context of self organized criticality [6], the conserved Manna model is well established in the literature by now [7, 8]. It is well known that the conserved Manna model shows an APT to infinitely many absorbing configurations when the density of the particles crosses a critical value  $\phi_c$ . There are several other model systems which show similar APT in presence of a conserved field, namely conserved lattice gas [3], conserved threshold transfer process [3] etc. All these models show similar critical behaviour in 2D and higher dimensions; in fact it has been conjectured that absorbing phase transitions in stochastic models with infinite absorbing states and activity coupled to a non diffusive conserved field define a unique universality class [3].

Recently, however, it has been shown [9] that conserved Manna model actually belongs to the DP universality class. The apparent non-DP behaviour was due to the use of random initial conditions, which take unusually long time to reach the stationary state in these models. It was also shown that random initial condition shows an ‘undershooting’ in the decay of activity  $\rho_a(t)$  which leads to erroneous determination of the critical point. This could be avoided by choosing suitable (natural) [9] homogeneous initial conditions, which reach the steady state in reasonably short time allowing one to approach closer to the critical point. Corresponding critical exponents are found to be in good agreement with DP values. These results motivate us to revisit APT in some recently studied one dimensional systems where the dynamics is very similar to that of the Manna Model, which also apparently show non-DP critical behaviour [10, 11].

The common features of these models [10, 11] are the presence of a conserved continuous local energy field, and an energy exchange dynamics; hereafter we refer to them as energy exchange models (EEM). The two versions of the EEM, maximal [11] and minimal [10] models differ

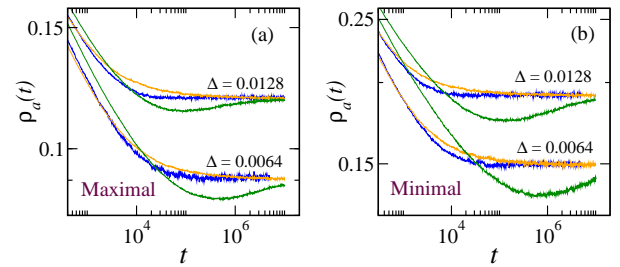


FIG. 1. (Color online) Undershooting: Comparison of  $\rho_a(t)$  versus  $t$  plot starting from random (green), deterministic (orange) and natural (blue) initial condition for both maximal (a) and minimal (b) models for a system of size  $L = 10^5$ . The undershooting becomes more pronounced as one approaches the critical density  $e_c$ .

in the way the activity is defined locally. In the maximal (minimal) version of the model a pair of neighbouring sites is called *active* when at least one of these sites has energy larger (smaller) than or equal to a predefined threshold value  $w$ . The *active* pairs reshare their total energy randomly among themselves dynamically. Previous studies of EEM have claimed that these models undergo an absorbing phase transitions characterized by critical exponents different from those of DP. It is noteworthy that the dynamics of these models are similar to that of the continuous conserved Manna model (CCMM), where an active site relaxes by distributing all its energy randomly among the neighbours. Since CCMM is known to have DP critical behaviour [9], it is natural to expect that EEM models might also belong to DP class. In particular we have observed that the activity density  $\rho_a(t)$  in EEM models shows an undershooting when the initial condition is random (see Fig. 1). As it is known [9] that undershooting in  $\rho_a(t)$  leads to erroneous estimation of the critical point, we avoid these ill effects in the EEM models by using a regular homogeneous initial condition. This leads to a more precise estimation of the critical point and critical exponents revealing that the true critical behaviour of APT in both maximal and minimal energy exchange models are in DP class.

**The Maximal Model:** The maximal version of energy exchange model is defined on a periodic one dimensional lattice with  $L$  sites labeled by  $i = 1, 2, \dots, L$ ; each site  $i$  containing a continuous energy  $E_i$ . A pair of neigh-

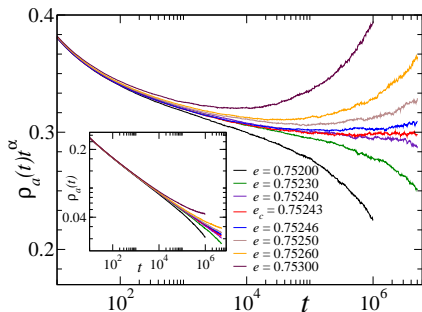


FIG. 2. (Color online) Critical point  $e_c = 0.75243$  for the maximal model is determined from  $\rho_a(t)t^\alpha$  versus  $t$ . The inset shows log scale plot of  $\rho_a(t)$  versus  $t$ ; the slope of  $e = e_c$  curve measured over last two decades corresponds to  $\alpha = 0.1598(1)$ .

bouring sites is called active when at least one of the sites have energy larger than or equal to a predefined threshold  $w$ . Otherwise, *i.e.* when both these sites have energy less than  $w$ , the pair is called *inactive*. The system evolves following a random sequential update rule where a randomly selected *active* pair  $(i, i+1)$  reshare their energies

$$\begin{aligned} E_i &\rightarrow r(E_i + E_{i+1}) \\ E_{i+1} &\rightarrow (1-r)(E_i + E_{i+1}). \end{aligned} \quad (1)$$

Here  $r \in (0, 1)$  is a uniform random number. Clearly this dynamics is energy conserving, *i.e.* the average energy  $e = \frac{1}{L} \sum_{i=1}^L E_i$  does not vary with time. Note, that similar conserving dynamics have been studied in different contexts of heat transport [12] and Econo-physics [13] earlier.

A configuration of the system where none of the pairs are active is called absorbing; it is easy to see that for  $e < w$ , the system has infinitely many absorbing configurations. However for  $e \geq w$  the system is active as there can not be any absorbing configuration for at least one of the sites must have  $E_i \geq w$ . This entails a possibility of a phase transition from absorbing to active phase when the average energy  $e$  is increased beyond some critical value  $e_c$ , which is smaller than the threshold value  $w$ . Without any loss of generality we proceed with  $w = 1$ . Any other arbitrary  $w$  would only scale the critical point by a factor  $w$ , *i.e.*  $e_c \rightarrow we_c$  as the dynamics is unaltered under this scale transformation.

The APT here is characterized by the average density of active pairs  $\rho_a = \langle \tau_i \rangle$ , where  $\tau_i = 1$  or 0 depending on whether the pair  $(i, i+1)$  is active or inactive. In the long time  $\rho_a(t)$  saturates to a steady value, which is non zero only in the active phase and serves as the orderparameter of this transition. It has been claimed [11] earlier that the critical behaviour of the model is different from DP. In the following we revisit the model carefully and show that the actual critical behaviour belong to DP. The primary reason for the apparent non-DP behaviour is possibly a small error in estimation of the critical point, which may be blamed to the use of random initial condition that shows undershooting.

The usual random initial conditions do not contain

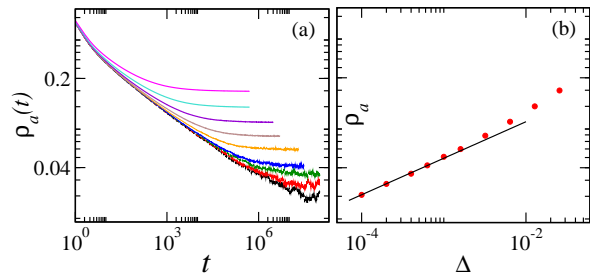


FIG. 3. (Color online) Estimation of  $\beta$  for the maximal model: (a) Plot of  $\rho_a(t)$  versus  $t$  for various energy density  $e > e_c$ . (b) Corresponding saturation values plotted against  $e - e_c$  in double logarithmic scale; the slope corresponds to  $\beta = 0.283$ .

the natural correlations existing in the stationary state. In particular the density profile in a random configuration is disordered as opposed to the stationary profile which is essentially flattened by the underlying diffusive dynamics. As discussed in Ref. [9], one should use the reactivated stationary configurations, called *natural initial conditions* for studying decay of activity in models with additional conserved field. However, generating natural initial condition takes substantial computational time which can be avoided in these continuous models by artificially creating a homogeneous density profile  $\{E_i\} = \{2e, 0, 2e, 0, \dots\}$ . This deterministic initial configuration is fully active with  $\rho_a = 1$  when  $e \geq 1/2$ . Fig. 1(a) shows a comparison of the decay of  $\rho_a(t)$  in the super critical regime starting from three different initial conditions : random (green), deterministic (orange) and natural (blue). Both the natural and deterministic initial condition do not show any undershooting as opposed to the random initial condition. In this work, we choose to study APT using deterministic initial condition as it is computationally inexpensive.

**Critical point and  $\alpha$  :** At the critical point  $e_c$  the activity decays as a power law

$$\rho_a(t) \sim t^{-\alpha}. \quad (2)$$

One can estimate the critical point  $e_c$  and exponent  $\alpha$  by plotting  $\rho_a(t)$  versus  $t$  for various values of  $e$  and looking for a power law decay. This estimate can be verified from the plot of  $\rho_a(t)t^\alpha$  against  $t$ ; the curve corresponding to  $e = e_c$  would remain constant in the long time limit. This procedure is illustrated in Fig. 2 which gives an estimate  $e_c = 0.75243(3)$ . The log scale plot of  $\rho_a(t)$  at  $e_c$  gives an accurate estimate of the critical exponent

$$\alpha = 0.1598(1).$$

This value of  $\alpha$  is a bit lower than the previous estimate 0.19 [11] and is in good agreement with the DP value (see Table I). The difference in  $\alpha$  here originates from the current estimate of  $e_c$  which is slightly higher than the previously reported [11] value 0.7508. Note that the system size used for the current work ( $L = 10^5$ ) is much larger. This correction of  $e_c$  is also expected to modify all the other critical exponents of the system substantially.

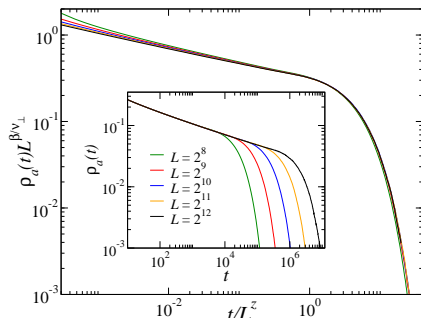


FIG. 4. (Color online) Finite size scaling for the maximal model :  $\rho_a(t)L^{\beta/\nu_\perp}$  for different  $L$  plotted against  $t/L^z$  collapse for  $z = 1.58$  and  $\beta/\nu_\perp = 0.26$ . The inset shows the corresponding unscaled data.

*Stationary state and  $\beta$*  : In the active phase the activity saturates to some finite value  $\rho_a$  which vanishes algebraically as one approaches the critical point,

$$\rho_a \sim (e - e_c)^\beta. \quad (3)$$

Here  $\beta$  is the order parameter exponent. Figure 3(a) shows  $\rho_a(t)$  as a function of  $t$  for various values of  $\Delta = e - e_c$  in the supercritical regime. Corresponding saturation values are plotted against  $\Delta$  in log-log scale (see Fig. 3(b)); the slope of the resulting straight line gives us an estimate

$$\beta = 0.283(4).$$

Again this value of  $\beta$  is much lower than the previous estimate [11] and is consistent with the DP value  $\beta_{DP} = 0.2764$ . Note that the present estimate is done for very small  $e - e_c < 10^{-3}$  which was not achieved earlier.

*Finite size scaling and  $z$*  : Now we turn our attention to the finite size scaling. For a finite system, the decay of  $\rho_a(t)$  at the critical point is expected to follow a scaling form

$$\rho_a(t) = L^{-\beta/\nu_\perp} G(t/L^z), \quad (4)$$

where  $z$  is the dynamic exponent. Thus, one expects that at the critical point  $e = e_c$ , activity density  $\rho_a(t)L^{\beta/\nu_\perp}$  for different values of  $L$  would collapse on to a single function when plotted against  $t/L^z$ . This data collapse, shown in Fig. 4 for systems of size  $L = 2^8, 2^9 \dots 2^{12}$ , yields an estimate

$$z = 1.58(2) ; \quad \frac{\beta}{\nu_\perp} = 0.26(1)$$

which are, again, in excellent agreement with the corresponding DP exponents.

In summary, all the critical exponents  $\alpha, \beta, z$  and  $\beta/\nu_\perp$  for the maximal model agree quite well with the corresponding DP values (see Table I). On the basis of these estimates we conclude that, contrary to the earlier claim, the maximal energy exchange model indeed belongs to the directed percolation class. The errors in the previously reported [11] exponents were caused by the inaccurate estimation of the critical point resulted from the use

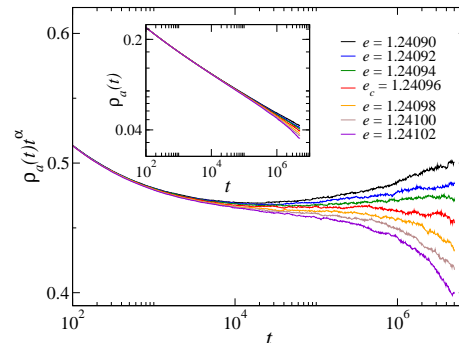


FIG. 5. (Color online) Critical point  $e_c$  for the minimal model :  $\rho_a(t)t^\alpha$  versus  $t$  curve for a system of size  $L = 10^5$  remains constant in the long time regime for  $e_c = 1.24096$ . The inset shows estimation of  $\alpha = 0.1595$  from  $\rho_a(t)$  vs.  $t$  plot.

of small system size and random initial condition (which suffers from undershooting).

**The Minimal Model** : In this section we revisit the minimal version of the model studied recently [10]. We restrict ourselves to one dimensional version of the model only. No particular difficulty, except the requirement of large computational time, is expected in studying the model in higher dimensions. Again the minimal model is known to have same mean field exponents as those of DP, the departure from DP, if at all there, should be prominent in one dimension (1D).

In the 1D minimal model, each site  $i$  has energy  $E_i$ , which is a continuous variable. A neighbouring pair of sites  $(i, i+1)$  is defined to be active when at least one of the sites  $i$  or  $i+1$  has energy less than or equal to a threshold value  $w$ . The active pairs in the model evolve according to the energy conserving dynamics (1). It is clear that *any* configuration having average energy  $e = \frac{1}{L} \sum E_i$  smaller than  $w$  is active. Configurations where all the sites have energy larger than  $w$  are absorbing; the system is likely to be trapped in an absorbing configuration for  $e \gg w$ . Thus, a transition from an active phase to an absorbing state is expected as the average energy is *increased* beyond some critical value  $e_c \geq w$ .

We must mention that the earlier study of the minimal model [10] was done at a fixed  $e = 1$  and varying the threshold  $w$ . This resulted in a critical point  $w_c = 0.810$ ; in other words in the  $(e, w)$  phase plane  $(1, w_c)$  is a critical point. It is evident that  $(\lambda, \lambda w_c)$  is also a critical point for any arbitrary  $\lambda$  as both the conditional statement  $E_i < w$  and the dynamics (1) are invariant under a scale transformation ( $E_i \rightarrow \lambda E_i, w \rightarrow \lambda w$ ). This implies that the critical line in the  $(e, w)$  phase plane is a straight line with unit slope passing through  $(1, w_c)$ . Clearly  $(1/w_c, 1)$  is also a critical point and it can be reached keeping  $w$  fixed; we choose to study the critical behaviour of the minimal model along the line  $w = 1$  by varying  $e$ , expecting the transition at  $e_c = 1/w_c$ .

We proceed further by defining the density of active pairs  $\rho_a$  as the order parameter. At the critical point  $\rho_a(t) \sim t^{-\alpha}$ . The decay of  $\rho_a(t)$  is studied starting

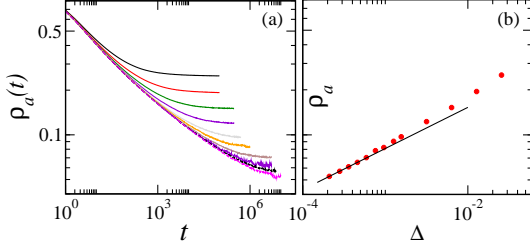


FIG. 6. (Color online) Estimation of  $\beta$  for the minimal model : (a)  $\rho_a(t)$  versus  $t$  for different  $e$  in the supercritical regime. (b) Corresponding saturation values as a function of  $\Delta = e_c - e$ ; the slope of the solid line corresponds to  $\beta = 0.2749$ .

from the spatially ordered homogeneous initial condition  $\{E_i\} = \{2e, 0, 2e, 0, \dots\}$ . Unlike random initial condition this homogeneous initial configuration does not suffer from undershooting (Fig. 1(b)).

Repeating the standard procedure for the determination of the critical point, discussed in section II, we plot  $\rho_a(t)t^\alpha$  versus  $t$  for different  $e$  in Fig. 5. This curve becomes flat at large  $t$  for

$$e_c = 1.24096(3).$$

The estimate of critical point is slightly higher than the previously reported value  $1/w_c = 1.23456$  [10]. Consequently, a careful study of the critical exponents in large system size shows a considerable departure from earlier estimates and agree remarkably with DP universality class.

The slope of the curve  $\rho_a(t)$  versus  $t$  in log scale (over last two decades) at the critical point  $e_c$ , shown in Fig. 5, gives an estimate of the exponent

$$\alpha = 0.1595(4),$$

which is in strikingly good agreement with  $\alpha_{DP}$ .

Next, we calculate the order parameter exponent  $\beta$  from the steady state density of activity  $\rho_a$ . Figure 6(a) shows the saturation of  $\rho_a(t)$  for various values of  $e$  ranging from 1.24080 to 1.21540. The stationary densities  $\rho_a$  are plotted against  $e - e_c$  in log scale in Fig 6(b) to obtain an estimate of  $\beta$ ; a linear fit for last two decades gives

$$\beta = 0.2749(8),$$

which, again, matches well with  $\alpha_{DP}$ .

The dynamical exponent  $z$  and  $\beta/\nu_\perp$  can be obtained from the finite size scaling, as at the critical point the activity density of finite systems scales as Eq. (4). In Fig. 7 we have shown the decay of  $\rho_a(t)$  (inset) for  $L = 2^8, \dots, 2^{12}$ . These curves have been collapsed onto a

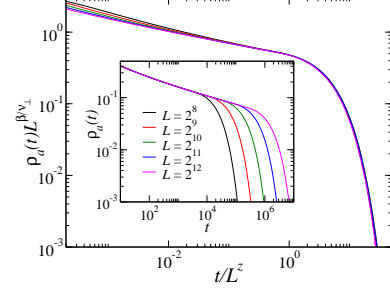


FIG. 7. Finite size scaling for the minimal model according to eq. (4). The best collapse corresponds to  $z = 1.54$  and  $\beta/\nu_\perp = 0.25$ . The inset shows the unscaled data.

single scaling function in Fig. 7 using

$$z = 1.54(2) \quad ; \quad \frac{\beta}{\nu_\perp} = 0.25(1).$$

Both the estimates are in good agreement with the corresponding DP values.

To summarize, the critical exponents of the minimal model also imply that, contrary to previous claims [10], this APT belongs to DP. We should mention that these differences in exponents are not in anyway related to the fact that the orderparameter in Ref. [10] was chosen differently. Instead of density of *active pairs*  $\rho_a$ , the average density of *sites* having  $E_i \leq w$  was used there. This order parameter  $\rho = \langle s_i \rangle$ , where  $s_i = 1$  ( $s_i = 0$ ) if  $E_i \leq w$  ( $E_i > w$ ), is related to  $\rho_a$  as

$$\rho_a = \rho + \langle s_i(1 - s_{i+1}) \rangle. \quad (5)$$

Since  $\rho_a$  is nonzero only when  $\rho \neq 0$  and they are dimensionally identical, the critical point and corresponding exponents would be same in both cases.

In conclusion, we have revisited absorbing phase transition in the energy exchange models in one spatial dimension. With extensive Monte-Carlo simulation on large systems, along with homogeneous initial conditions, we establish that both the maximal and the minimal version of the model belong to DP universality class; the critical exponents are summarized in Table I.

With this study we want to emphasize the fact that one has to be very careful in exploring absorbing phase transition in presence of a conserved field. The conserved field imposes additional constraints on the evolution of the system and introduces a long time scale. Random initial conditions may need very long time to saturate, producing unwanted transient effects like undershooting. These undesirable features may lead to erroneous estimation of critical point and hence inaccurate critical exponents. One must take care of these ill effects while studying APT in a system with a conserved field.

Model	$e_c$	$\alpha$	$\beta$	$z$	$\beta/\nu_\perp$
Maximal	0.75243(3)	0.1598(1)	0.283(4)	1.58(2)	0.26(1)
Minimal	1.24096(2)	0.1595(4)	0.2749(8)	1.54(2)	0.25(1)
DP [1]	-	0.15946	0.27648	1.5802	0.2520

TABLE I. Comparison of critical exponents.

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